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Avicennan Infinity: A Select History of the Infinite through Avicenna

1. Introduction

Puzzles arising from the notion of infinity have been part of the philosophical landscape since virtually the beginning of Western philosophy. One needs merely think of Zeno's famous dichotomy paradox with its infinity of halfway points. Yet denying the infinite is equally paradoxical. One needs merely think of numbers, for if they were not infinite it would seem that there would be some greatest number to which the unit one cannot be added. Moreover, banish the infinite and one banishes irrational numbers, such as pi, as well as continuous magnitudes. It was the appearance of puzzles such as these and others that motivated Aristotle to explore the notion of the infinite in the detail that he did in both his *Physics* and *De Caelo*[1]. Moreover, it was Aristotle's analysis of the infinite that would dominate much of the subsequent discussion of the topic in the ancient world. Thus claims such as 'an actual infinite is impossible'[2] and 'the infinite cannot be traversed'[3] — both claims that Aristotle makes and uses throughout his physical writing — came to be treated as virtual dogma by many subsequent philosophers. Despite the authority of Aristotle, the medieval Muslim philosopher Avicenna (980-1037) in some way denied both of these points as well as others that involved Aristotle's conception of the infinite. In fact, I shall argue that as a result of certain modifications that Avicenna makes in clarifying what infinity is, he develops the concept to a point that it can no longer be thought of as a strictly Aristotelian infinity, but must be viewed as an Avicennan infinity. I shall further argue that many of the influences going into Avicenna's new re-conception of infinity had their origins in the Arabic-speaking intellectual world of medieval Islam.

[1] Cf. Aristotle, *Physics*, III, 4-8 and *De Caelo*, I, 5-7.

[2] Cf. *Physics*, III, 5, where Aristotle presents the *aporiai* associated with the infinite, all of which he thinks arise only when the infinite is taken to be actual, and his explicit statement to this effect at III, 7, 207b11-12.

[3] Cf. Aristotle, *Physics*, III, 4, 204a1-6 and *De Caelo*, I, 5, 272a3.

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To make good on these two theses, I begin by briefly considering Aristotle's theory of the infinite and then John Philoponus' criticism of certain Aristotelian theses based on Aristotle's own conception of infinity. I then turn to four developments concerning the infinite prevalent in the medieval Islamic milieu. These are (1) the emergence of a new non-Aristotelian form of argumentation against the possibility of an actual infinite ; (2) defenses and new attitudes towards the possibility of an actual infinite ; (3) a suggestion that the infinite is traversable ; and (4) a discussion of infinity's relation to number. In the final section I conclude by showing how Avicenna modifies and incorporates all of these influences into a conception of infinity and continuity that is greater than the sum of its parts, and is a truly unique Avicennan theory.

2. The Infinite in the Classical Greek World : Aristotle and Philoponus

Aristotle formally defined the infinite (Grk. *to apeiron* ; Arb. *lā mutanāhī*) as « that of which, [whatever] the quantity taken is, there is always something [more] outside to take »[4]. In addition to this formal definition, he also just as readily identified the infinite with that which cannot be traversed (Grk. *oun dioristeon* ; Arb. *mā lā- yumkina š-šurū*ʿ *fīhi*)[5]. He furthermore distinguished between an actual infinite and a potential infinite[6]. Very loosely an actual infinite involves some definite or complete infinite quantity all parts of which exists at the same time, as, for example, the way that some have conceived the expanse or magnitude of the universe, whereas a potential infinite involves an incomplete process that lacks an end or limit. Aristotle also identified two forms of potential infinities : infinity by addition, such as, for example, the series produced by the n+1 functor, 1, 2, 3 ... n, and a potential infinity by division, such as the series produced by the 1/2n functor, 1/2, 1/4, 1/8 ... 1/2n. Either series can proceed infinitely, and yet at no time will there ever be all the elements constituting the series.

While Aristotle happily admitted potential infinities, he adamantly denied that an actual infinity could ever exist. His arguments for this conclusion, in

[4] See Aristotle, *Physics*, III, 6, 207a7-8.

[5] Cf. Aristotle, *Physics*, III, 4, 204a1-6.

[6] For various contemporary discussions of Aristotle's conception of the infinite see J. Hintikka, *Aristotelian Infinity*, « Philosophical Review », 75, 1966, pp. 197-218 ; J. Lear, *Aristotelian Infinity*, « Proceedings of the Aristotelian Society », 80, 1975, pp. 187-210 ; W. Charlton, *Aristotle's Potential Infinites*, in L. Judson ed., *Aristotle's* Physics, Oxford University Press, Oxford 1995, pp. 129-149 ; and J. Bowin, *Aristotelian Infinity*, « Oxford Studies in Ancient Philosophy », 32, 2007, pp. 233-250.

both his *Physics* and *De Caelo*, are what might best be termed 'physical proofs'. Roughly such argumentation takes the form that if there were an actually infinite body (or even an actually infinite space), motion — which is the proper subject matter of natural philosophy — would be impossible[7]. That is because if there were an actual infinite, there could be no way to identify a natural place, such as an absolute up or down, for a natural place is some determinate limit towards which a given body naturally moves and at which it naturally is at rest. An actual infinite however lacks any determinate limit. If there were no natural place, continues Aristotle, no body could naturally move to its natural place. Since Aristotle delimited the elements exclusively in terms of their natural motions — the element earth, for example, is that which naturally moves downward, while the element fire is that which naturally moves upward — the impossibility of natural places implies the impossibility of the elements themselves. Inasmuch as one or more elements constitute natural bodies, that is, the subjects undergoing the motion that is studied by natural philosophy, the elements are principles of natural philosophy. Thus to countenance an infinite is to do away with the very possibility of natural philosophy or physics. In the *De Caelo*, Aristotle further adds arguments to show that the (apparent) diurnal motion of the heavens would likewise be impossible if the cosmos were infinite[8]. One such argument begins with the then commonly held belief that the heavens rotate around the Earth. As such any given arc described by two radii extending from the center of the universe (namely the Earth) should be traversed during this heavenly rotation. Now since the arc between the radii of an infinite body when considered at an infinite distance from the center would likewise be infinite, and an infinite cannot be traversed, the heavens' rotation, if they were infinite, would be impossible. Thus Aristotle concludes, if there were an infinite body, both sub-lunar and supra-lunar motion would be impossible. Yet clearly both exist, and thus the natural philosopher must reject the notion of an actual infinite as a principle of his science.

Let this overly brief summary of Aristotle's conception of the infinite suffice. The key points are that, while Aristotle believed that the existence of potential infinities, understood as never ending processes, is obvious, he adamantly denied that an actual infinite could ever exist, otherwise, its existence

[7] Cf. ARISTOTLE, *Physics*, I, 5, 204b35 ff. and *De Caelo*, I, 6, 273a7-21. Alternatively, Aristotle also sometimes argues that if there were an infinite body it would have infinite weight, but since weight, at least according to Aristotle, is one of the factors determining speed, an infinite body would move with infinite speed, a conclusion that Aristotle finds absurd ; see *De Caelo*, I, 6, 273a21 ff.

[8] Cf. ARISTOTLE, *De Caelo*, I, 5, 271b26 ff.

would render the science of physics impossible. Additionally, he seemed to take it simply as a first principle that it is impossible for an infinite — however understood, whether as actual or potential — ever to be fully traversed.

Despite Aristotle's groundbreaking treatment of the infinite, it was not without its share of difficulties, particularly when conjoined with other Aristotelian doctrines, such as the belief that the world had no temporal beginning. These difficulties came to a head at the end of the classical period in the polemical writings of the Christian Neoplatonist, John Philoponus (ca. 490-570). Philoponus criticized a number of central theses of Aristotelian science, but perhaps none more strenuously than the thesis of the world's past eternity[9]. Moreover, his critique of this Aristotelian position used Aristotle's own principles concerning the infinite against him — such principles as an actual infinite is impossible and an infinite cannot be traversed[10].

Philoponus had two main lines of objection : one, an eternal world would entail that an actual infinite has come to exists and so an infinite has been traversed, and, two, there would be sets of infinities of different sizes, and so sets larger than that beyond which there is nothing more[11]. The first of Philoponus' arguments takes the following form : if the world were eternal, as Aristotle believed, then it would entail that there has been an infinite number of past days, but if during those infinite days, one human per day, for example, were born, then an actually infinite number of humans would have come to exist. It does no good to say that they do not all exist right now, since the *number* of humans who have existed must be actually infinite. Aristotle himself, however, had said that not even number considered separately can be infinite[12]. Moreover, the world's past eternity runs afoul, criticized Philoponus, of the Aristotelian dictum that an infinite cannot be traversed, for the world would have gone through an infinity of days (as well as thing generated during those days), but again traversing the infinite is impossible absolutely. The challenge of this critique — at least for those philosophers who maintained the eternity of the world as Avicenna would — is to show what, if any,

[9] For an overview of many of Philoponus' anti-Aristotelian theses see R. SORABJI ed., *Philoponus and the Rejection of Aristotelian Science*, Duckworth, London 1987.

[10] For a discussion and summary of Philoponus' specific uses of the Aristotelian doctrines of infinity against Aristotle see R. SORABJI, *Infinity and the Creation*, in ID., *Philoponus and the Rejection of Aristotelian Science* cit., pp. 164-178, especially at p. 170.

[11] Philoponus usually presents both arguments together as a couplet. See *Contra Proclum*, ed. H. RABE, Teubner, Leipzig 1899, I 3, pp. 8.27-11.21 and XVIII 3, pp. 619.3-620.19. Also see *Contra Aristotelem*, fragment 132. While Philoponus' *Contra Aristotelem* is not longer extant, Christian Wildberg has gathered together the existing fragments in *Philoponus, Against Aristotle on the Eternity of the World*, trans. C. WILDBERG, Cornell University Press, Ithaca, NY 1987.

[12] Cf. ARISTOTLE, *Physics*, III, 5, 204b.

meaningful sense might be given to the existence of an actual infinity, and in what sense, if any, the infinite can be traversed.

Philoponus' second complaint is that if the world were eternal, there would be varying sizes of infinities, and indeed the infinite would be susceptible to increase. For example, if the universe's existence were extended infinitely into the past, then the Sun, Moon and all the planets would have orbited the Earth an infinite number of times (at least based on ancient and medieval cosmology, which has the Earth at the center of the universe) ; however, Saturn makes a rotation around the Earth once ever 30 solar years ; Jupiter once every 12 solar years ; Mars once every 2 solar years and the Sun, of course, once every year. Consequently, for example, the Sun must have made 30 times as many rotations around the Earth as Saturn has. Thus asks Philoponus, « If it is not possible to traverse the infinite once, then how is not beyond all absurdity to assume ten thousand times the infinite, or rather the infinite an infinite number of time ? »[13]. The challenge that this paradox presents, then, is how to make sense of different 'sizes' of infinities. In modern terms, we might ask, 'Is there any sense to the notion of infinities with different cardinalities ?'.

With this bare-bones presentation of Aristotle and Philoponus, we have (I believe) touched on most of the salient points and controversies surrounding the infinite (at least as it was treated by natural philosophers) that the medieval Arabic-speaking world would inherit from the classical Greek world. We now turn to certain further developments involving the notion of infinity that were taking place within the Islamic medieval milieu.

3. The Infinite in the Medieval Islamic World : Philosophers and Mathematicians

Within the medieval Islamic world, one of the most obvious points of departure from the purely Aristotelian understanding of the infinite was how philosophers argued against the possibility of an actually infinite magnitude. Again Aristotle's preferred approach was 'physical proofs'. In contrast, many of the philosophers writing in Arabic preferred what might best be described as a 'mathematical-style proof'. A version of this mathematical-style argument already appeared in a fully developed form as early as the first Arabic philosopher, al-Kindī (ca. 801-866)[14]. While structurally the argument is quite similar to arguments for different theses found in Proclus' *Institutio Physica*[15],

[13] *Against Aristotle on the Eternity of the World* cit., p. 146.

[14] For a translation and commentary of al-Kindī's argument see N. Rescher, H. Khatchadourian, *Al-Kindi's Epistle on the Finitude of the Universe*, « Isis », 56.4, 1965, pp. 426-433.

[15] See *Procli Diadochi Lycii Institutio Physica*, ed. A. Ritzenfeld, Teubner, Leipzig 1912.

which itself is a piece of Aristotelian natural philosophy structured on the *Elements* of Euclid, al-Kindī's proof against the existence of an infinite is neither found in that work nor in any of Aristotle's physical works and their subsequent commentaries (at least as far as I can see). In al-Kindī's proof, he presents a series of mathematical axioms, and then through a series of steps employing only those axioms *and* the hypothesis that an actually infinite magnitude exists, he generates a series of contradictions. Since the axioms are seemingly self-evident, al-Kindī concludes that the supposition of an actually infinite magnitude is what gives rise to the contradictions, and so is impossible. Since the argument was to become a mainstay for discussions of the infinite in the medieval Arabic-speaking world, and also given that Avicenna would avail himself of a variant of the proof when treating the infinite, we should consider it in some detail.

Al-Kindī prefaces his version of the proof with the following initial assumptions (IA):

(IA-1) if two finite magnitudes are joined, then the resulting magnitude is finite;
(IA-2) the finite is not infinite;
(IA-3) if one magnitude is larger than another, then the latter is smaller than the former;
(IA-4) if there are two magnitudes and one is smaller, then the smaller magnitude is smaller than the larger magnitude, but equal to a portion of the larger magnitude;
(IA-5) if two magnitudes are equal, then the dimensions between the two bodies' limits are equal;
(IA-6) the infinite is without limits, whereas the finite is limited;
(IA-7) if a magnitude is added to one of two equal magnitudes, then the enlarged magnitude is both (a) larger than the other magnitude and (b) larger than it originally was.

Al-Kindī next asks us to imagine an infinite magnitude. For ease of exposition, imagine a ray that extends from the Earth infinitely into space. Now imagine that some finite amount were separated from the ray, for example, the amount extending from the Earth to the edge of our galaxy. In this case the amount of what continues on beyond the edge of our galaxy must be either finite or infinite.

First step: If that remaining amount that extends beyond our galaxy is finite, argues al-Kindī, then should the finite portion that was separated be rejoined to the finite portion that remains, the two together would be finite (from IA-1). The two together were supposedly infinite by the initial hypothesis.

Thus the resultant would be finite and infinite (from IA-2), which is a contradiction. So what remained after the initial separation could not be finite.

Second step: if what remains is infinite, then should the finite portion that was removed be rejoined to the infinite portion that purportedly remained, the Original infinite magnitude, *O*, must be either larger than or equal to the infinite portion Remaining after the separation, *R*. *Sub-step 1*: If *O* is larger than *R*, then *R* is smaller than *O* (from IA-3), and so *R* is equal to a portion of *O* (from IA-4). If *R* is equal to a portion of *O*, however, then the dimensions between the limits of *R* and a portion of *O* are equal (from IA-5). In that case, though, *R* is limited (for we are talking about *the dimensions between* R*'s limits*), but that which is limited is finite (from IA-6). Thus *R* is finite, but it was assumed to be infinite. Thus there is again a contradiction. *Sub-step 2*: Assume that *O* is equal to *R*, in which case, a part of *O*, namely, what corresponds with *R*, is equal to the whole, namely, *O*, but the part is not equal to the whole (from IA-7). Therefore, *O* is not equal to *R*. From the initial assumption of step two — namely that the original (purportedly) infinite magnitude must be either larger than or equal to any infinite magnitude assumed to remain after some finite portion is separated off from that original magnitude — and the conclusions of the two sub arguments — namely that the original magnitude can be neither larger than nor equal to the magnitude resulting from the separation — al-Kindī can conclude by simple *modus tollens* that what remained after the separation was not infinite. What remained after the separation, again, had to be either finite or infinite, but both assumptions in conjunction with one or more of the initial axioms entailed a contradiction. Something must go, and what must go, concludes al-Kindī, is the assumption that an actual infinite is possible.

The questionable step in the argument is sub-step 1 and the definition of 'equal' used in IA-5, namely, 'if two magnitudes are equal, then the dimensions between the two bodies' *limits* are equal', for this account of equality already begs the question against the possibility of there being any actual infinite, which again is the very issue at stake. To make the question-begging status of the definition clear consider instead of a single infinite magnitude of which one part is removed and then replaced, two equal and infinite magnitudes that are parallel to one another. Given IA-5, one could immediately infer that since the two infinite magnitudes are equal to one another, the dimensions between their limits must be equal (at least according to the understanding of 'equal' in IA-5). Thus, the two magnitudes are limited, but that which is limited is finite, a contradiction. Here the defenders of an actual infinity need merely assert that the notion of equality at play is applicable to only finite or limited magnitudes, and so has no bearing on the infinite. Equality may mean (and in fact does mean) simply that an equivalence relation holds between two things, namely, the relation between them is reflexive, symmetrical and transitive. In the case of this latter definition of equality there is no mention of limit, and thus al-Kindī's alleged contradiction never arises.

Despite this flaw in the argument — a step, I might add, that does not appear in later philosophers' versions of the proof — the general strategy involved in it was to have a far-reaching appeal[16]. In particular, we see several later Arabic-speaking natural philosophers employing the strategy of removing some finite portion from some posited infinite magnitude and then making comparisons. Indeed it is primarily a variation of this argument, and only secondarily (if even that) Aristotle's physical proofs, that Avicenna employs when he argues that there cannot be a material or spatial instantiation of an actual infinity.

While al-Kindī believed that the above argument could be used to show that any and all magnitudes must be finite — whether continuous magnitudes, such as size of the universe, or discrete ones, such as the number of days that the universe has existed — most subsequent natural philosophers took the argument to show merely that a material or spatial instantiation of an actual infinity was impossible. According to these later natural philosophers the argument was not at all applicable to past days, which were viewed as constituting a potential infinity rather than an actual one. Moreover, among certain medieval mathematicians who applied the principles of their discipline to problems of natural philosophy, the issue of whether there could be an actual infinity or whether an infinity could be traversed were still being debated. Thus in a short series of questions addressed by one Abū Mūsá ʿĪsá Usayyid to the Ṣābian mathematician and astronomer Ṯābit ibn Qurra (d. 901) the first query is whether the number of souls (presumably human souls) are finite or infinite[17]. The very presence of the question at least suggests that the issue of an actual infinity was being debated at his time, and so was an alive topic as early as the late ninth century.

Although the bare-bones statement of the problem as presented in Ṯābit's treatise — namely, 'Are souls finite or not ?' — provides no details behind the

[16] In addition to Avicenna, see, for example, Ibn Bāǧǧa, who provides a version of it in terms of numbers rather than magnitudes in his *Physics* commentary [IBN BĀǦǦA, *Šurūḥāt As-Samāʿ aṭ-ṭabīʿī*, ed. M. ZIYĀDA, Dār al-Kindī, Beirut 1978, p. 36.14-19] ; Ibn Ṭufayl, who repeats the argument in his philosophical novel *Ḥayy ibn Yaqẓān* (« Alive Son of Awake ») [IBN ṬUFAYL, *Hayy Ben Yaqdhân*, ed. L. GAUTHIER, Imprimerie Catholique, Beirut 1936, pp. 76-77 (Arabic), pp. 58-59 (French)] ; and as-Suhrawardī, who offers his own take of the argument in *Ḥikmat al-Išrāq* (« The Philosophy of Illumination ») [SUHRAWARDĪ, *Ḥikmat al-išrāq*, J. WALBRIDGE, H. ZIAI eds., Brigham Young University Press, Provo, UT 1999, p. 44].

[17] See A. I. SABRA, *Ṯābit Ibn Qurra on the Infinite and Other Puzzles, Edition and Translation of His Discussion with Ibn Usayyid*, « Zeitschrift für Geschichte der arabisch-islamischen Wissenschaften », 11, 1997, pp.1-33, esp. [1].

question, the philosophical puzzle may have been one that Avicenna would mention approximately a hundred years later and is perhaps best known from al-Ġazālī's *Tahāfut*[18]. The issue at stake in this puzzle concerns the mutual compatibility of the doctrines of the world's eternity and the soul's immortality when coupled with the Aristotelian dictum that an actual infinity is impossible. In its simplest form the argument takes as its primary premises (1) the world has existed an infinite number of days in the past ; (2) the human soul is immortal ; and (3) an actual infinite is impossible[19]. Premises (1) and (2) seem to mutually imply that an actual infinity of human souls should presently exist, but an actual infinity is impossible by (3). Therefore, one of the premises must be false. If this is in fact the issue at stake in Ṯābit's treatise, then it becomes immediately clear why the question of the finitude or infinitude of souls might be immediately relevant.

In fact there is at least some reason for thinking that this puzzle may have had its origin in Philoponus and thus may have been the topic of Ṯābit inquiry, for all of its basic premises do appear in Philoponus' commentary on *De anima*[20]. Unfortunately, Philoponus' *De anima* commentary seems not to have been known in the Arabic world (or at least an-Nadīm does not mention it in his *Fihrist*). Moreover, nothing like the above argument is found in any of the extant fragments of Philoponus' *Contra Aristotelem* — of which at least six sections, according to an-Nadīm, were known in the medieval Arabic-speaking world — or in his *Contra Proclum*. Similarly, the argument is not found in the extant Arabic synopsis of Philoponus' arguments against the eternity of the world[21]. In short, while it is my suspicion that Philoponus was the avenue by

[18] For a discussion of Avicenna's presentation of the argument, which we shall discuss more fully below, as well as its subsequent history see M. E. MARMURA, *Avicenna and the Problem of the Infinite Number of Souls*, in ID., *Probing in Islamic Philosophy, Studies in the Philosophies of Ibn Sina, al-Ghazali and Other Major Muslim Thinkers*, Global Academic Publishing, Binghamton, NY 2005, pp. 171-179. Also see AL-GHAZĀLĪ, *The Incoherence of the Philosophers*, ed. and trans. M. E. MARMURA, Islamic Translation Series, Brigham Young University Press, Provo, UT 1997, «First Discussion : first proof », §§30-32.

[19] To be truly demonstrative the argument would in fact need some further subsidiary premises. For a detailed presentation of the argument as it appears in al-Ġazālī and Thomas Aquinas see G. MASSEY, *St. Thomas Aquinas on the Age of the Universe : Pious Advocate or Self-Interested Partisan ?*, « Divinatio », 24, 2006, pp. 67-97.

[20] For the text see *Commentaire sur le* de Anima *d'Aristote [par] Jean Philopon*, ed. G. VERBEKE, Publications Universitaires, Louvain & B. Nauwelaerts, Paris 1966, p. 16, 83-95 and p. 38, 90-6 ; English translation in *On Aristotle on the Intellect (*de Anima *3.4-8)*, trans. W. CHARLTON, Cornell University Press, Ithaca, NY 1991.

[21] See S. PINES, *An Arabic Summary of a lost work of John Philoponus*, in *The Collected Works of Shlomo Pines*, *Studies in Arabic Versions of Greek Texts and in Mediaeval Science*, vol. II, The Magnes Press, Jerusalem 1986, pp. 294-326.

which this argument concerning an infinity of souls came into the medieval Islamic milieu, such a thesis at this point can neither be proved nor disproved.

Whatever the case though the issue of whether there could be an infinity of souls was clearly being debated as early as the late ninth century. Ṯābit's own position was that, despite the received view, there could be, and in fact are, actual infinities. He gives as two examples Gods' knowledge of the infinite number of individuals who have existed in the past, presently exist or will exist in the future, as well as God's knowledge of the infinite species of geometrical figures and the infinite species of numbers, all of which presumably exist as actual infinities of objects of knowledge in the mind of God. One would find these examples of actual infinities convincing however only if one likewise believes that the corresponding intelligibles all exist as distinct in the divine mind. Such an assumption, however, would not have been shared by most of the philosophers working in the Neoplatonized Aristotelianism of the *falsafa* tradition, who instead held that the Divinity knows a plurality of things solely by knowing its simple essence as the ultimate cause of those things[22]. To populate the divine mind with a plurality of intelligible objects or ideas jeopardizes divine simplicity, which most medieval philosophers — Jews, Christians and Muslims alike — took to be central to a proper philosophical theology. Whatever the case, what is important for our purposes is that thinkers in the pre-Avicennan period were entertaining the idea of the possibility of an actual infinity and particularly an infinite number of souls.

In addition to the admittance of actual infinities and a resolution of purported paradoxes involving them, some of these mathematicians were also attempting to establish the possibility of traversing an infinite. One such thinker was Abū Sahl al-Qūhī, who was working during the second half of the tenth century. Al-Qūhī's strategy, which follows the method of projections, was to describe a situation where a light source, which is directed at a post so as to produce an infinitely long shadow, moves around a semicircle in a finite period of time, and yet the spatial magnitude covered by its shadow's motion neither has a first beginning point nor last ending point[23]. Since there is neither a first starting point nor a last ending point, the distance covered by the shadow is without limits and so is infinite. Again, however, the time it takes for the shadow to cover that distance is equal to the finite period of time it takes for the light source to cover the finite distance described by the

[22] See for instance, AVICENNA, *Metaphysics*, ed. and trans. M. E. MARMURA, Brigham Young University Press, Provo, UT 2005 (Islamic Translation Series), VIII.7.

[23] For a detailed description of al-Qūhī's thought experiment see R. RASHED, *Al-Qūhī* vs. *Aristotle : On Motion*, « Arabic Sciences and Philosophy », 9, 1999, pp. 7-24.

semicircle. Thus, al-Qūhī would conclude that an infinite distance (the distance covered by the shadow) is traversed in a finite period of time (the time taken by the motion of the light source). Although al-Qūhī's thought experiment tacitly assumes that there is an actually infinite space corresponding to the distance covered by the shadow — an assumption that all of those working in the Aristotelian tradition of natural philosophy would have rejected — the very fact that the issue of traversing an infinite was being discussed *and* being discussed in physical terms (light sources and shadows) strongly suggests that many of Aristotle's theses concerning the infinite were under discussion and open to dispute.

Even within the *falsafa* tradition, one sees interesting developments related to the infinite based solely upon Aristotelian principles. For example, the Christian Baghdad Peripatetic Yaḥyà ibn ʿAdī dedicated an entire treatise to the infinite with the purpose of demonstrating, on Aristotelian grounds, that infinity simply could not be predicated of number[24]. In short, he wanted to show that the notion of a numeric infinity was conceptually incoherent, and thus many of the paradoxes associated with infinity were in fact the result of philosophical confusion. His general strategy is to argue that there is no particular number that is infinite (for while all the particular numbers exhaust numbering, no particular one of the infinity exhausts numbering), and so infinity is not predicated of any particular number[25]. If there are no particulars however, he continues, then neither is there some universal that subsumes them under it (for among physical things the universal is the form of a given thing stripped of matter and free of accidents that is conceptualized in the mind). So infinity is not predicated of any universal[26]. Consequently, if infinity is not predicated of any particular number nor is there a universal infinity predicated of number, then infinity is not predicated of number absolutely. Infinity, Ibn ʿAdī goes on, is predicated only of that to which number does not belong or apply at all. To make this point, he gives the following example : 'Not white' might be truthfully predicated of a body in two ways — one, if the body is transparent and so simply has no color, and two, if the body is colored, but not white. In both cases, it is true to say of the body that it is 'not white'. Infinity is predicated of something in a manner analogous to the first way. Thus being infinite is said of whatever, not because it has an infinite number, but precisely because whatever it is has *no number* at all.

[24] YAḤYÁ IBN ʿADĪ, *The Philosophical Treatises, A Critical Edition with an Introduction and Study*, ed. S. KHALĪFĀT, University of Jordan, Amman 1988, « Treatise on the Infinite », pp. 135-140.

[25] *Ibid.*, pp. 135-136.

[26] *Ibid.*, pp. 136-137. Here Ibn ʿAdī gives Aristotle's *Categories* 5, 2b1-4 as a proof text : « Again color in body is equally in *a* body, for if it is not in one of the particulars, it itself would not be in body».

All in all, Ibn ʿAdī's treatise is a defense of Aristotle against « many of our brethren currently dabbling in philosophy ... [who] persist in the doubt that arises through bad reasoning on the part of imagination, and the (erroneous) belief about number that it has some limit corresponding with its beginning while having no limit corresponding with its end »[27]. While it cannot be proven conclusively, Ibn ʿAdī may very well have intended this work to be a response to Philoponus' critique of Aristotle. Certainly Ibn ʿAdī's use of 'our brethren' (*iḫwāninā*) can in the present context suggest 'fellow Christians', such as Philoponus, although it can equally mean 'fellow philosophers', albeit dilettantes in the field by Ibn ʿAdī's reckoning. Still, it was Philoponus' expressed opinion that while the number of days must have some beginning or first, there would be no last day, again the very thesis that Ibn ʿAdī wants to refute[28]. Furthermore, Ibn ʿAdī thinks that by interpreting Aristotle in such a way that infinity does not belong to number, he has also resolved all the puzzles pushed upon Aristotle involving things that are more and less infinite, for, he argues, more and less apply essentially and primarily to discontinuous quantities such as numbers, whereas infinity, as he has just argued, in no way applies to number[29]. He then specifically gives a puzzle reminiscent of those found in Philoponus :

> « Since [the puzzle] is imagined as arising precisely on the part of number, one group, owing to their supposition that days are infinite and likewise years are infinite — where, since every year has three hundred and sixty-five and a quarter days, the number of days is clearly more than the number of years — imagined that they had found an infinite number more than an infinite»[30].

'An infinite number more than an infinite' is precisely the absurdity to which Philoponus believed an eternal past gave rise. So there is at least some textual evidence that Philoponus and those using Philoponus' arguments were Ibn ʿAdī's intended target.

Still, based solely on Ibn ʿAdī's text, it cannot be ruled out that he may also have had in mind mathematicians such as Ṯābit ibn Qurra. For Ibn ʿAdī additionally raises the following puzzle, which he thinks that his own analysis of infinity does away with :

> « Also because individuals, that is, a species (sing.) of species (pl.), are infinite in their number, they (erroneously) believe that it was possible to add to its number

[27] Yaḥyá Ibn ʿAdī, *The Philosophical Treatises* cit., p. 135.
[28] See *Contra Proclum* cit., XVIII 3, pp. 619.3-620.19.
[29] Yaḥyá Ibn ʿAdī, *The Philosophical Treatises* cit., p. 138.
[30] *Ibid.*, pp.138-139.

> (that is, the infinite) a number of individuals of another species. In that case, the number of individuals of the two species, which is infinite, would become more than the number of the individuals of the single species, which is infinite »[31].

If the species (*anwāʿ*) in question here are species of numbers or geometrical figures, then the argument to which Ibn ʿAdī is responding does seem close to an argument that Ṯābit ibn Qurra suggested for why there must be infinities of varying size[32].

Whoever the target, Ibn ʿAdī's analysis of the infinite seems designed to prevent paradoxes that arise from treating infinity as a number or species of number. Thus it very well may have been Ibn ʿAdī's intention to remove the notion of infinity altogether from the mathematical sciences. At the very least Ibn ʿAdī's position renders many specifically mathematical intuitions and paradoxes generated from those intuitions inapplicable when stated in terms of infinity.

With this as a brief background to the philosophical developments involving the notion of infinity as that notion appeared in the pre-Avicennan period of Arabic philosophy, let us turn to Avicenna.

4. Avicennan Infinity

Avicenna treats the infinite in all three of his encyclopedic works — the *Šifā'*, the *Naǧāt* and *al-Išārāt wa-t-tanbīhāt*[33] — as well as at least two smaller treatises — a letter to the Vizier Abū Saʿd and another work dealing with whether the past has a temporal beginning[34]. While there can be no doubt that

[31] Yaḥyá Ibn ʿAdī, *The Philosophical Treatises* cit., p. 139.

[32] Ṯābit's argument is that odd numbers and even numbers — which we know to share an equivalence relation — are equally infinite, and yet either set is only half of the set of all natural numbers. Moreover, there are an infinite number of sets of infinities of differing sizes, for the numbers divisible by three are infinite, and yet the set of these numbers has one-third the members as the set of natural numbers. The same goes, *mutatis mutandis*, for numbers divisible by four, five and so on *ad infinitum*. What makes Ṯābit's argument here particularly interesting for the history of mathematics is that he clearly recognizes our modern definition of infinity as a set capable of being put into one-to-one correspondence with a proper subset of itself. See *Ṯābit Ibn Qurra on the Infinite and Other Puzzles* cit., pp. 24-25, [13]-[14].

[33] See *aš-Šifā', As-Samāʿ aṭ-ṭabīʿī*, ed. J. al-Yāsīn, Dār al-Manāhil, 1996, III.7-11, pp. 210-227 ; *an-Najāt*, ed. M. Dānišpaẓūh, Dānišgāh-yi Tihrān, Tehran 1985, Part IV (« Physics »), treatise 2, section xi, pp. 244-252 ; *al-Išārāt wa-t-tanbīhāt*, ed. J. Forget, E. J. Brill, Leiden 1892, *namaṭ* I, pp. 94-95.

[34] The first treatise can be found in *Lettre au vizir Abû Saʿd*, Editio princeps *d'après le manuscrit de Bursa, traduction de l'arabe, introduction, notes et lexique*, ed. and trans. Y. Michot, Les Éditions al-Bouraq, Beirut 2000 (Sagesses musulmanes, 4). The second treatise, which still exist only in manuscript form — Brit. Lib. Or. 7473, Ahmet III 3447, and Nuruosmaniye 4894 — is currently being edited and translated by David C. Reisman.

Avicenna knew and drew upon Aristotle's account of infinity, it is equally obvious that Avicenna was aware of the dangers as well as the developments involving the infinite. In the remainder of this study, I want to look at how Avicenna appropriated and adapted both Greek and Arabic sources to shape an Avicennan notion of infinity.

We have already noted that Aristotle formerly defined the infinite as « that which when anything is taken from it, there is always something quantitatively beyond it that can be [further] taken »[35], but we have also seen that he equally described it as that which cannot be traversed. Indeed, Philoponus would latch onto this latter understanding of the infinite to argue against Aristotle's belief concerning the age of the world. For, complained Philoponus, if the existence of the world did not come to be at some definite moment in the finite past, then an infinite would have been traversed, whether an infinity of days or generated things. Such a conclusion, however, contradicted Aristotle's own dictum that the infinite cannot be traversed.

In contrast, I know of no place in the Avicennan corpus where Avicenna outright denies that an infinite can be traversed absolutely[36]. Instead, he consistently adopts an understanding of the infinite more in line with Aristotle's formal definition. So, for example, in his letter to the Vizier Abū Saʿd, Avicenna defines the infinite as « a quantity or something possessing a quantity such that anything you take from it you also find something of it different from what you took and you never reach something beyond which there is nothing of it »[37]. As for traversing the infinite, if Avicenna mentions its impossibility at all, it is always in a qualified way : an infinite cannot be traversed *in a finite period of time*[38]. Without this qualification, Avicenna saw

[35] This translation is slightly different from the one given previously, since I have here translated *Physics*, II, 6, 207a7-8 from the Arabic translation of Aristotle ; *Arisṭuṭālīs, aṭ-Ṭabīʿī*, ed. ʿA. Badawī, 2 vols., The General Egyptian Book Organization, Cairo 1964/65.

[36] It is worth nothing that in the whole of the *Physics* of the *Šifā'* Avicenna mentions the traversal of an infinite only (as far as I can see) three times : once with the qualification that the infinite cannot be traversed *in a finite period of time* (III.4), once when presenting an argument of Aristotle's (III.8), and then again in the same place to distance himself from the proposition that an infinite cannot be traversed.

[37] *Letter to Vizier Abû Saʿd* cit., p. 28.

[38] Even this seemingly self-evident claim Avicenna felt compelled to prove, which he did by considering the proportional ratios of a number of different interrelated factors associated with motion, such as the force impressed by the mover, the weight of and resistance to the moved object, as well as the time, distance and speed of the motion. His general strategy was then to show that if any one of these variables is set at ∞ then the others would either likewise go to ∞ or to 0. Thus an infinite distance's being traversed in some (positive) finite period of time contradicts the proportionalities that Avicenna felt he had demonstrate to exist between the various arguments of the functions that he has considered. See, for example, the *Physics* of the *Šifā', as-Samāʿ aṭ-ṭabīʿī* cit., III.10 and IV.15.

no problem with an infinite's being traversed, again provided that there is an infinite amount of time to traverse it. While such a position clearly puts Avicenna's understanding at odds with such mathematicians as al-Qūhī concerning the way that an infinite is possibly traversed, the difference between them is again not primarily because the one thinks that traversing an infinite is impossible absolutely; rather, it is because, as we shall see, Avicenna has reason to believe that infinitely extended spatial magnitudes are impossible, whereas al-Qūhī seems to take their existence for granted.

As for the possibility of traversing an infinite, there is in fact limited agreement between the philosopher and mathematician. So, for instance, in the *Ilāhiyyāt* of his *Šifā'* as part of a response to Philoponus, Avicenna quite explicitly maintains that not only is it possible to traverse an infinite temporal causal chain, but it is in fact necessary.

> « We do preclude an infinite [number of] ancillary and preparatory causes, one [temporally] preceding the other. In fact, that must necessarily be the case, since each temporally created thing has become necessary after not having been necessary because of the necessity of its cause at that moment ... and its cause also having become necessary. So with respect to particular things, there must be an infinity of antecedent things by which the actually existing causes necessarily come to be certain actual causes of [the particulars] »[39].

Here Avicenna is explaining why a given temporal event or thing comes to be at the time that it does and not earlier, where the reason is that the matter was only prepared to take on a new form at that time[40]. As such there must have been temporally prior causes that prepared the matter, but of course those temporally prior causes are also temporal events or things, which themselves need temporally prior causes, and so on *ad infinitum*. Thus, according to Avicenna, an infinite number of temporally prior preparatory causes must have been traversed.

Recall however that Philoponus had a follow-up objection, namely that the traversal of an infinite, even if all the members are not currently present, still entails that an actual infinite has been realized, and an actual infinite, no matter how construed, is impossible, or at least Philoponus would have one believe. Avicenna's rejoinder, at least as it appears in the *Šifā'* is to claim that Philoponus fails to appreciate the distinction between *each one* (*kull wāḥid*) and *whole* (*kull*)[41]. So, for example, while it is true that *each one* of the parts

[39] *Metaphysics* cit., VI.2, p. 202.7-10.

[40] Avicenna provides the details, which are only implicit in the cited quotation from the *Metaphysics*, in his *Physics*; see *as-Samā' aṭ-ṭabī'ī* cit., III.11.

[41] See *as-Samā' aṭ-ṭabī'ī* cit., III.11, pp. 226-227.

of a thing is a part, it is false that the *whole* of that thing is a part. Similarly, contends Avicenna, while it is possible that each one of an actual infinite has existed, it need not be possible for the whole of that infinite to exist as a whole.

In fact, using the each one/whole distinction Avicenna makes a move parallel to that of Yaḥyà ibn ʿAdī. Again recall that Ibn ʿAdī's strategy for dealing with mathematical paradoxes involving infinity was to deny that infinity belongs to numbers, and thus the paradoxes never arise. Avicenna makes a less striking but isomorphic move, now however in terms of sets (sing. *jumla*)[42]. Avicenna argues that the whole of past events is not, as it were, collected together into an actually existing set. At best, he observes, they have been collected together in some intellectual depiction (*waṣfu l-ʿaql*). A collection in an intellectual depiction, however, is only equivocally like a collection existing in reality or extra-mentally, which is a genuine set, for the collection *all animals* as a logical notion existing in the intellect, Avicenna points out, is « decidedly not the set of them [existing extra mentally] »[43]. Of course, if something does not exist, then it is inappropriate to say that it is *actually* any thing, at least in any proper sense of 'actual'. Thus concludes Avicenna it is simply unforgivable to speak of the set of past events as *actually* infinite, for no such set exists. The parallel between Avicenna's argument here and Ibn ʿAdī's earlier argument becomes apparent once it is observed that the standard ancient and medieval understanding of number was in terms of a collection or set of units[44]. Thus Avicenna's denial that there exists any set of past events to which infinity belongs is simply a more qualified version of Ibn ʿAdī's point that infinity does not apply to number. Where Avicenna's claim differs from Ibn ʿAdī's is that Avicenna's at least allows the possibility of an infinite set and so number. It is just that in the case of past (or future) events there is no actual set of which infinities can be predicated.

Using the same strategy, Avicenna further addresses Philoponus' objection about the rotations of the planets in that they would be more or less infinite. Again there is no actually existing infinite set of rotations ; rather, Avicenna reminds us that they are said to be infinite in that « whatever number our estimative faculty imagines to belong to the motions, we find a number that was before it »[45]. As for the whole set of rotations or the like, that does not

[42] The response that Avicenna provides seems to be hinted at by Aristotle in *Physics*, III, 8, 208a20-21, as well as having some similarities with Simplicius' response, even though it does not seem as if Simplicius' commentary on the *Physics* was available in Arabic. See SIMPLICIUS *In Aristotelis Physicorum*, ed. H. DIELS, typis et impensis G. Reimer, Berlin 1882, pp. 494.14-495.5.

[43] See *as-Samāʿ aṭ-ṭabīʿī* cit., III.11, p. 226.

[44] Cf. ARISTOTLE, *Physics*, III, 7, 207b7 as well as *Metaphysics*, X, 1, 1053a30 and XIII, 9, 1085b22 ; also see EUCLID, *Elements*, book VII, definition 2. Although also see AVICENNA, *Metaphysics* cit., III.3, pp. 80 ff. where he argues that number (*ʿadad*) cannot strictly be defined in terms of a multiplicity of units.

[45] *As-Samāʿ aṭ-ṭabīʿī* cit., III.11, p. 226.

exist. Now, continues Avicenna, notions such as 'more' and 'less' as well as 'finite' and 'infinite' either apply or do not apply to non-existent things. If they do not apply to non-existent things, then the objection disappears, whereas if such terms do apply, then they equally apply to the infinity of future rotations that will occur. Since Philoponus' in fact conceded that future time will be infinite, he finds himself hoisted on his own petard[46].

Of course Philoponus' original objections remain viable if one simply rejects the possibility of infinities outright — whether actual or potential, past or future — as certain *mutakallimūn* seemed to have done[47]. Thus one may simply posit it as self-evident that no infinite exists and no infinite can be traversed under any conditions. In general, Avicenna is extremely leery of such *a priori* claims. For instance, he recounts a public debate concerning the infinite and the continuous between himself and one Abū l-Qāsim al-Kirmānī in which he remarks about al-Kirmānī :

> « Whenever he was forced into at tight place, [al-Kirmānī] said, "My intellect only immediately accepts this" and sometimes he said, "I have been inspired to this", but he is heedless that [even] the intellects of competent people vary with respect to that spontaneity »[48].

For Avicenna, the ability or inability of the estimative faculty to imagine something simply is no sure guide as to whether that thing is a real possibility or not[49]. What one needs is a demonstration. Thus the question arises : What can one demonstrate about the existence or non-existence of the infinite.

Like al-Kindī before him, Avicenna sets to one side Aristotle's physical-style proofs when arguing against the existence of an actual infinite, and instead prefers a variation of al-Kindī's mathematical-style argument. Loosely Avicenna's argument, which appears in all three of his encyclopedic works, runs thus : Imagine, for example, two absolutely rigid beams, that is, something

[46] *As-Samāʿ aṭ-ṭabīʿī* cit., III.11, p. 226.

[47] While it may seem extreme to deny that even a potential infinite exists, there were arguments for such a position. So, for example, if something truly is potential, then its being actualized is possible, and so if there were a potential infinite, an actual infinite must be possible. Yet most medieval thinkers admitted that the existence of an actual infinite is impossible. Consequently, a potential infinite must be impossible too, and so the distinction between a potential and actual infinity is vacuous. See, for example, Ibn Mattawayh, cited in A. Dhanani, *The Physical Theory of Kalām, Atoms, Space, and Void in Basrian Muʿazilī Cosmology*, E. J. Brill, Leiden 1994, p. 151

[48] *Letter to Vizier Abû Saʿd*, p. 31.

[49] In *as-Samāʿ aṭ-ṭabīʿī* cit., III.11, p. 226., II.9 and again at IV.8 Avicenna makes this point quite explicitly.

that cannot give way so as to stretch, and moreover suppose that these beams extend from the Earth infinitely into space. Next imagine that some amount, *x*, is removed from one of them, again, for instance, the distance from the Earth to the end of our galaxy, and call that beam from which *x* has been *R*. Now imagine that *R* is pulled to the Earth and *R* is compared with the beam from which nothing had been removed, call that original beam *O*. In this case Avicenna thinks that either one of two situations must hold.

> « On the one hand, the two [beams] inasmuch as they extend together might [exactly] map onto [one another] in extending such that the greater and the lesser are equal, which is absurd. On the other hand, [*R*] might not extend as far, but fall short of the other, [*O*] and so [*R*] is finite, but the extra part [*x*] was also [finite]. Thus the composite is finite, and so the original [beam] was finite »[50].

Again since the beams are absolutely rigid, *R* could not have stretched so as to extend the extra length *x*. Consequently, *R* must be less than *O* by a length equal to *x*. Since the two beams are lying side-by-side, *R* must either absolutely map onto *O* so as to extend along with it infinitely into space, or fall short of *O*. If *R* falls short of *O* on the side extending into space, then where it falls short of *O* is a limit of *R*, in which case *R* is limited on both the side extending into space and on the Earth side. In that case, *R* is finite, but it was assumed to be infinite. So there is a contradiction. If *R* does not fall short of *O*, but exactly corresponds with *O*, then *R* is not less than *O*, but it was posited that *R* is less than *O* by the length *x*, and so again there is a contradiction.

The same line of reasoning also works if the magnitude is unlimited on all of its sides, since one can simply posit a limit in the magnitude by supposition. Thus, for example, if the space to one's left and right were infinitely extended, then one could posit the point where one is as the limiting point and then apply the original argument. Likewise, claims Avicenna, the argument works in the case of the « actually [infinite] essentially ordered number ». Avicenna perhaps is thinking of something like a number line composed only of integers that extends infinitely, where each integer is represented by some unit magnitude, in which case, again, the original argument applies. Since Avicenna's argument involves mapping one magnitude onto another, let us refer to it as 'the mapping argument'.

There are a few things to note about Avicenna's presentation of the mapping argument. First, nowhere does he use the troublesome definition of 'equal' that rendered al-Kindī's version of the argument invalid. Second, and again unlike al-Kindī, Avicenna does *not* believe that the argument applies to

[50] *Najāt*, II.2, « On the Finite and the Infinite » cit., p. 244.

all magnitudes. Thus in the *Naǧāt*, where Avicenna offers what is arguably his most daring and innovative analysis of infinity, he limits the argument merely to showing that

> « there does not arise a continuous magnitude that exists *essentially possessing an infinite position* (*aḏ-ḏāt ḏū waḍʿ*) and equally there is no *essentially ordered* number that exists infinitely all together, where I mean by 'essentially ordered' that part of it is naturally and *per se* prior to other parts »[51].

In the *Šifā'* he makes the same point: « it is impossible that there exist as wholly actualized some infinite magnitude, number or [set of] numbered things *having an order either in nature or position* (*waḍʿ*) »[52]. Avicenna consistently uses 'position' (*waḍʿ*) as one of the concomitants that follows upon matter, and in fact in the version of the proof as it appears in *al-Išārāt wa-t-tanbīhāt*, Avicenna make clear that the argument merely proves that « corporeal extension (*al-imtidād al-ǧismāniyy*) must be finite »[53]. In it simplest terms the mapping argument, Avicenna seems to think, merely shows that there can be no material instantiation of an actual infinite.

Again, Avicenna sees his argument as precluding the possibility of the existence of any *actually infinite essentially ordered quantity*; however, since as we shall see, he does allow for the existence of certain actual infinities, it will be helpful to get clear on exactly what an *actually infinite essentially ordered quantity* entails. Avicenna suggests two conditions that a quantity would need to meet in order to be an actually infinite essentially ordered quantity. The first condition is that it would be a quantity that absolutely cannot be more or less. Thus Avicenna writes, « that which is infinite in this way is that which when it exists and then is assumed to be susceptible to increase and decrease, an absurdity necessarily follows »[54]. Avicenna's description of an actual infinite in his letter to the vizier Abū Saʿd, helps shed further light on this point, for there he characterizes the actual infinite thus: « An actual infinite is described as that which has now extended infinitely to the limit beyond which there is no limit »[55]. We might think of an actually infinite quantity, anachronistically, as the set of quantitative elements consisting of members who all exist at once, and, in addition, there is no further like element that can be included within the set or class. In effect,

[51] *Najāt*, II.2, « On the Finite and the Infinite » cit., p. 244.
[52] *As-Samāʿ aṭ-ṭabīʿī* cit., III.8, p. 212.
[53] *Al-Išārāt wa-t-tanbīhāt, namaṭ* cit., I, p. 95.
[54] *Najāt*, II.2, « On the Finite and the Infinite » cit., p. 245.
[55] *Letter to Vizier Abû Saʿd* cit., p. 33.

Avicenna is simply repeating the condition we have already seen him raise when treating infinities of varying size : 'more and less' and 'finite and infinite' can only be predicated of sets all of whose members exist together. Since Avicenna explained this in terms of 'each one' and 'whole', let us call this condition the 'wholeness condition'.

The second condition that Avicenna notes is that the members of this set or class must be essentially ordered. We have see Avicenna's understanding of an essential order as he presents it in the *Nağāt*, namely, as that whose parts vis-à-vis one another have a natural and essential priority relation. Here I take Avicenna to mean by 'essentially ordered' something like a mathematical sequence, where every member has a well-defined placement in the order such that some function can be given that uses either the preceding members or a member's position in the sequence to identify the placement of any particular member. This is obvious in the case of natural numbers, for the number 3, for instance, cannot come just any where in the sequence of natural numbers, but must come after 2 and before 4. The same also holds for continuous magnitudes, for one can use some arbitrary metric, such as a system of meters, to assign numbers to conventionally determined segments of some definite positive amount that, as it were, constitute that continuous magnitude, as, for example, a certain number of unit-meters constitute the length of a hallway or the like. In that case, the conventionally determined segments of the magnitude can be treated like 1 meter, 2 meter, 3, meter, and so are analogous to counting numbers. Again, the general idea is that each member in the set has some well-defined position relative to the other members, such that the set can be called 'ordered'. Let us call this condition the 'ordering condition'.

In summary, Avicenna's mapping argument precludes only the existence of something with infinite members who meet *both* the *wholeness condition* and the *ordering condition*, that is, a set (or at least what can be treated like a set in the case of continuous magnitudes) whose members (1) all exist at one and the same time *and* (2) have a sequential ordering. What Avicenna in effect has done is to give a precise understanding of what type of actual infinite demonstrably *cannot* exist at least insofar as the mapping argument is concerned. Should there be some actual infinite that fails to meet one or both of the aforementioned conditions, then the impossibility of its existence has not been demonstrated.

In fact, like Ṯābit ibn Qurra before him, Avicenna believes that there are actual infinities ; however, unlike Ṯābit, Avicenna does not believe this based upon an appeal to unanalyzed intuitions about the objects of God's knowledge. Quite the contrary, Avicenna argues that there exist at least two types of actual infinities precisely because there exists infinities that fail to meet one or the other of the two conditions mentioned above. Moreover, argues Avicenna, not only are these two types of actual infinities not demonstrably impossible, they in fact can be positively demonstrated to exist. Thus Avicenna observes :

> « As for either [(1)] when all the members are infinite and not existing all together, but are in the past and the future, nothing prevent their existing one before or after another, while not existing all together ; or [(2)] when the number itself has neither a positional nor natural ordering, then nothing prevents them from existing all together. There is no demonstration that [these types of actual infinities] are impossible. Quite the contrary there is a demonstration that they exist. As for the first kind, time and motion prove that it is such. As for the second kind, a sort of infinite number of virtuous and vicious intellects[56] (as the case will become clear to you) proves it to us. All of this is susceptible to increase and its being susceptible to [increase] is of no use in applying the mapping [argument] because what is not ordered, either in nature or position, will not be susceptible to [such] mapping, and that which does not exist all together is even further removed with respect to it. The entire way that most people go about disproving the infinite in the past is from either highly held common [opinions] or sophistical premises, but none of them are demonstrative »[57].

It would take us well beyond the scope of the present paper to provide Avicenna's 'demonstrations' for either the eternity of time and motion[58], or the immortality of the human intellect[59], but suffice it to say that he thought he himself had demonstrated these positions, whereas he thought that the absolute denial of an unqualified actual infinite could not be demonstrated.

Thus an infinite past with its accompanying infinite rotations of the Sun, Moon and planets, which gave rise to one of Philoponus' objections to the eternity of the world, is a token of the first type of actual infinite, namely, one that fails to meet the wholeness condition. An actually infinite number of immortal intellects, which I have suggested may have been another argument in Philoponus' arsenal of proofs for a temporal creation, is a token of the second type of actual infinite, namely, one that fails to meet the ordering condition.

[56] Literally 'angels' and 'demons' ; Michael Marmura, who mentions this argument in his article *Avicenna and the Problem of the Infinite Number of Souls* cit., p. 173, fn. 17, notes that Avicenna uses 'angels' and 'demons' often to refer to the good and bad souls.

[57] *Najāt*, II.2, « On the Finite and the Infinite » cit., pp. 245-246.

[58] For a presentation of his argument for the eternity of time see J. McGinnis, *Time to Change : Time, Motion and Possibility in Ibn Sīnā*, in the *Proceedings of International Ibn Sīnā Symposium, Istanbul, May 2008* (forthcoming).

[59] See T.-A. Druart, *The Human Soul's Individuation and Its Survival after the Body's Death : Avicenna on the Causal Relation between Body and Soul*, « Arabic Sciences and Philosophy », 10, 2000, pp. 259-273.

An infinite number of immortal intellects fails to meet the ordering condition precisely because the disembodied intellects have no natural or positional order amongst themselves. In other words, since immortal intellects are disembodied, they do not exist in spatially discrete places such as to form a line, such as people waiting to buy tickets for a movie. Similarly they have no essentially ordered numeric sequence such that one can, for instance, arrange Peter's intellect before or after Paul's intellect, such as the alphabetical arrangement of names on a roster. Perhaps one could provide a *temporally* sequential ordering for these intellects, as for example, Peter was born *before* Paul ; however, this ordering does not presently exist among them, such as to make the mapping argument applicable. Moreover such a sequential ordering seems to come down to the ordering of an infinite past, which, again, Avicenna thinks is unproblematic.

In the remainder of the chapter on the infinite in the *Naǧāt* Avicenna further clarifies how the two permissible types of actual infinities exist, and says that even though their members are actually infinite, these two are also still potential infinities as well. In other words, the infinities in question cannot be treated as closed sets, but must be treated as open ended, namely, as involving an on-going process. Thus they are actually infinite in that were one to try to count out the number of past days or presently existing intellects, one would never reach a limit beyond which there is nothing of the same kind to be counted. Quite the contrary, at any particular number one reached there would always be something more to count. They are potentially infinite in that this actual infinity is always in process of having new members added to it. Avicenna sums up this point in the following passage :

> « What is infinite is always an actually existing thing, namely, from the perspective that it never reaches a given limit that is its final limit, for what exists of it always is described as that which has not yet reached another limit or the limit beyond which there is no limit. What is infinite is always potential, that is, something of its nature is away potential. This is the future, whereas its existence in the past had no beginning in the past, but there was one after another since it was, and were you to begin counting them from now, the counting would not stop at some terminal point »[60].

Here, then, one has an actually existing infinite in that no assignable particular number designates all of the present objects, for there will always be more objects beyond any assigned particular number. Yet one also has a potentially existing infinite in that new objects are constantly in the process of being added to the presently existing infinity of objects.

[60] *Najāt,* II.2, « On the Finite and the Infinite » cit., p. 249.

Let me conclude here and finish by way of summary. For Aristotle, the infinite was a topic proper to natural philosophy and as such required physical-style proofs. Among the proofs offered in his *Physics* were purported demonstrations that any actual infinity is absolutely impossible. Additionally, Aristotle, and a long tradition following him, viewed it as virtually a truism that an infinite cannot be traversed. Drawing upon these assumptions the Christian Neoplatonist, John Philoponus, criticized a number of Aristotelian physical theses, such as the eternity of the world. Such a thesis, argued Philoponus, violated a number of Aristotelian dicta concerning infinity, as, for example, the impossibility of an actual infinity being realized, an infinite's being traversed, the ability to increase an infinite as well resulting in infinities of varying cardinality, and, entailing the present existence of an actual infinity, namely, of human souls. In the Arabic-speaking world, we see a shift away from a number of these traditional Aristotelian tenets involving infinity: philosophers and mathematicians alike began applying mathematical approaches to the physical questions involving the infinite. So, for example, al-Kindī provides a mathematical style proof against the possibility of an infinite. Also mathematicians in the medieval world of Islam were themselves more willing to challenge the Aristotelian consensus. Thus, based upon examples involving numbers, figures and mathematical proofs, these thinkers challenged the dogma that either an actual infinite is impossible or that an infinite could not be traversed. Others, such as Ibn ʿAdī seemed inclined in an opposite direction, and wanted to limit the application of mathematical analyses to the infinite, thus rendering a number of the paradoxes raised by Philoponus null.

Avicenna in developing his own understanding drew upon all of these sources. Thus following Ibn ʿAdī, Avicenna too wanted to eliminate the paradoxes that arose when one treated infinity like a number; however, unlike Ibn ʿAdī, Avicenna found a way of doing this without denying the application of infinity to number absolutely. Indeed, Avicenna in many ways preferred the clarity that a mathematical treatment of the infinite provided over the merely physical-style proofs that Aristotle favored. In fact, it is arguably the case that the willingness of mathematicians to call into questions such seemingly self-evident propositions as 'an actual infinite is impossible' and 'an infinite cannot be traversed' that emboldened Avicenna to adopt just these views, while doing so in a qualified way that did not appeal to mathematical abstracta, which would have been suspect to most natural philosophers, but to counter-examples and demonstrations drawn from physics proper. In this respect, perhaps Avicenna's most important innovation is his careful analysis of 'actual infinity', and the recognition that many of the theorems of finite quantities simply could not apply to infinite ones. It is here that one sees the emergence of a truly Avicennan infinity.

ABSTRACT

Puzzles arising from the notion of infinity have been part of the philosophical landscape since virtually the beginning of Western philosophy. Aristotle in his *Physics* undertook a careful analysis of it, primarily in physical terms, and John Philoponus used that same analysis to undermine a number of tenets of Aristotelian physics. While the notion of infinity still remained a central notion of natural philosophy in the medieval Islamic intellectual milieu, the exploration of the infinite increasingly began being done in more purely mathematical terms with both physicists and mathematicians leading the way. After a very brief survey of the infinite in the classical Greek world, this study considers some of the innovations in the theory of the infinite at the hands of philosophers and mathematicians working in Arabic, and then concludes with Avicenna's synthesis of these various advancements into an uniquely Avicennan infinity.

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